

COPS AND ROBBERS, GAME THEORY AND ZERMELO'S EARLY RESULTS

ATHANASIOS KEHAGIAS AND GEORGIOS KONSTANTINIDIS

ABSTRACT. We provide a game theoretic framework for the game of *cops and robbers* (CR). Within this framework we study certain assumptions which underlie the concepts of *optimal strategies* and *capture time*. We also point out a connection of these concepts to early work by Zermelo and D. König. Finally, we discuss the relationship between CR and related pursuit games to *reachability games*.

1. INTRODUCTION

In this note we study the game of *cops and robbers* (CR), first introduced in [11, 12]. Our goals are the following.

- (1) We examine the concepts of *optimal strategies* and *capture time* from a *game theoretic* point of view. While this is a natural formulation, it is generally not used in the CR literature. Notable exceptions are [7, 5, 6] but, in our opinion, these papers overlook some important details. We must stress that we consider [7, 5, 6] extremely valuable contributions, which introduce novel concepts and *reach correct conclusions*. However, we believe that some issues underlying these conclusions have not been analyzed completely.
- (2) Similar issues have been discussed in two early papers by Zermelo [14] and D. König [9], in the context of *chess*. This brings us to our second goal: bring to the attention of CR researchers the connection between Zermelo (an early game theorist), D. König (an early graph theorist) and CR (a natural meeting point between graph and game theory).
- (3) Our final goal is to present a connection between CR (and related pursuit games) and *reachability games* [3, 10]. This connection has been, to some degree, anticipated in the CR literature but never explicitly noted.

We assume the reader is familiar with CR as described in [11, 1]. We follow the notation and terminology of [7]. We will assume the CR game is played by one cop and one robber on a *cop-win* graph $G = (V, E)$ (extension to robber-win graphs and multi-cop games is straightforward but omitted, in the interest of brevity). We denote the cop's (resp. robber's) move at the t -th round by x_t (resp. y_t).

2. GAME THEORETIC ANALYSIS OF CR

Discussion of (*time*) *optimal strategies* has appeared relatively recently in the CR literature. Two examples are [7, 5] and we will take these as representative of the approach prevalent in the CR literature. We will also examine a more recent discussion, which appears in [6].

While the concept of strategy is central in the CR literature, it is usually introduced informally, without providing a precise definition. Regarding optimality, in [7] a strategy is called “optimal

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for the cop if no other strategy gives a win in fewer moves” and a strategy is “optimal for the robber [...] if no other strategy forces a longer game”. This definition is also rather informal.

We will provide more rigorous definitions of “strategy” and “optimality” in *game theoretic* terms. To this end we introduce the following definitions.

- (1) A *game position* is a triple (x, y, p) , where x (resp. y) is the current cop (resp. robber) position and p is the player whose turn it is to play.
- (2) A *history* is a sequence of cop and robber moves. A *finite* history has the form $x_0y_0x_1y_1\dots x_t$ or $x_0y_0x_1y_1\dots y_t$; an *infinite* history has the form $x_0y_0x_1y_1\dots$.
- (3) A *legal cop* (resp. *robber*) *strategy* is a function s_C (resp. s_R) which maps finite histories to *legal* next cop (resp. robber) moves:

$$x_{t+1} = s_C(x_0y_0\dots x_t y_t) \in N[x_t] \quad (\text{resp. } y_{t+1} = s_R(x_0y_0\dots y_t x_{t+1}) \in N[y_t]).$$

A cop strategy s_C also provides a cop move $x_0 = s_C(\emptyset)$ at the beginning of the game, when presented with the *empty* history \emptyset .

- (4) A *memoryless* (legal) cop strategy is one which only depends on the current cop and robber position; in other words: $\forall x_0y_1\dots x_t y_t : s_C(x_0y_1\dots x_t y_t) = s_C(x_t y_t)$. Similarly, a *memoryless* (legal) robber strategy satisfies: $\forall x_0y_1\dots y_t x_{t+1} : s_R(x_0y_1\dots y_t x_{t+1}) = s_R(y_t x_{t+1})$.
- (5) A *play* is a history h which either terminates with a *capture* (i.e., $h = x_0y_0x_1\dots y_{t-1}x_t$ with $y_{t-1} = x_t$ or $h = x_0y_0x_1\dots x_t y_t$ with $x_t = y_t$) or, if no capture takes place, continues for an infinite number of rounds (i.e., $h = x_0y_0x_1\dots$ with $y_{t-1} \neq x_t$ and $x_t \neq y_t$ for all t). Since a pair (s_C, s_R) fully determines the corresponding play h , we will denote the length of the play by $T(s_C, s_R)$.
- (6) When the cop uses s_C and the robber uses s_R , the robber’s (resp. cop’s) *payoff* is $T(s_C, s_R)$ (resp. $-T(s_C, s_R)$). CR with this payoff is a *two-person, zero-sum* game [8].

We always have $\sup_{s_R} \inf_{s_C} T(s_C, s_R) \leq \inf_{s_C} \sup_{s_R} T(s_C, s_R)$. If we actually have

$$(1) \quad \sup_{s_R} \inf_{s_C} T(s_C, s_R) = v = \inf_{s_C} \sup_{s_R} T(s_C, s_R),$$

then we say that v is the *value* of the game. Suppose (1) holds and there exists a cop strategy s_C^* (resp. a robber strategy s_R^*) such that

$$(2) \quad \forall s_R : T(s_C^*, s_R) \leq v, \quad (\text{resp. } \forall s_C : T(s_C, s_R^*) \geq v)$$

then we say that s_C^* (resp. s_R^*) is an *optimal strategy*. If optimal strategies (s_C^*, s_R^*) exist, then we also have [8]

$$(3) \quad T(s_C^*, s_R^*) = \sup_{s_R} \inf_{s_C} T(s_C, s_R) = \inf_{s_C} \sup_{s_R} T(s_C, s_R).$$

(In the CR context, $T(s_C^*, s_R^*)$ is commonly called *capture time* [4].) It is well known [8] that (3) holds for all *finite* games (i.e., they have both a value and optimal strategies). But CR is an *infinite* game (it may last an infinite number of rounds and, consequently, there is also an infinite number of strategies) hence (3) must be *proved*. To this end, it suffices to prove the following.

Lemma 2.1. *If G is cop-win, there exists a number \overline{T}_G and a cop strategy \overline{s}_C such that $\sup_{s_R} T(\overline{s}_C, s_R) \leq \overline{T}_G$.*

In other words: if the cop can effect capture in a finite number of rounds, he can do so in a *bounded* number of rounds, and the bound is *independent of the robber strategy*. Some readers

may consider this obvious, but we will soon argue that proving Lemma 2.1 is not trivial. At any rate, using the lemma, we can prove the following.

Theorem 2.2. *For every cop-win graph G there exist memoryless cop and robber strategies σ_C^* , σ_R^* such that the CR game played on G has value $T(\sigma_C^*, \sigma_R^*)$ (which we call capture time).*

The proof of Theorem 2.2 is straightforward and hence omitted. The basic idea is that, since the cop strategy $\bar{\sigma}_C$ effects capture in at most \bar{T}_G rounds, *for every robber strategy*, it suffices to examine the “truncated” game \bar{CR} in which the robber wins (and receives infinite payoff) if he can avoid capture for \bar{T}_G rounds. Since \bar{CR} is a finite game, it has a value and both cop and robber have memoryless optimal strategies.

Furthermore, Theorem 2.2 can be generalized as follows: when K cops and one robber play on a graph G with *cop number* $c(G) \geq 1$, optimal memoryless cop and robber strategies exist for every $K \geq 1$ (note: if $K < c(G)$ then capture time is infinite and *every* cop strategy is optimal). Once again, details are omitted.

In the definitions leading to Lemma 2.1 and Theorem 2.2 we have followed a “standard” game theoretic approach; this, as already noted, is generally not used in the CR literature. A notable exception is the formulation presented in [6], which is very similar to the one we have used, with one important difference. Namely, the definition of “strategy” used in [6] is the same as our definition of “memoryless strategy”; in other words, it appears that the authors of [6] *only consider memoryless strategies*. We will argue in Section 4 that this choice is based on an important implicit assumption.

3. BOUNDEDNESS OF CAPTURE TIME

Apparently Lemma 2.1 has been considered so obvious that it has been used without proof. In fact, the following stronger assumption has been used [5]:

(4) *“If the cop has a winning strategy he can play so that no state of the game is repeated.”*

“Game state” is another term for “game position”. Lemma 2.1 follows immediately from (4): since the number of possible positions (excluding those of the 0-th round) is $2|V|^2$, if the cop can win without repeating any game position, then he can win in at most $|V|^2$ rounds.

We will now argue that (4) is correct but its proof, while short, is not trivial.

It turns out that an analog of (4) has been claimed by Zermelo for the game of *chess*. Both chess and CR are two-person, zero-sum games of perfect information, in which the players move alternately; furthermore, the number of positions is finite in both games; finally, Zermelo studied a version of chess in which the “*draw on three repeated moves*” rule is not applied, hence the game can last an infinite number of rounds. Because of the close analogy between CR and chess, Zermelo’s original statements and later revisions are highly relevant to our discussion.

All the passages quoted in the rest of this section come from the excellent paper [13], which (among other things) discusses Zermelo’s [14] and D. König’s [9].

In 1913 Zermelo wrote [14], in which he studies (in the context of chess) the following question: “given that a player is in ‘a winning position’, how long does it take for White to force a win?”. His answer is the following (the similarity of the following passage to (4) is obvious).

Zermelo claimed that it will never take more moves than there are positions in the game. His proof is by contradiction: Assume that White can win in a number of moves greater than the number of positions. Of course, at least one winning position must have appeared twice. So White could have played at the first

occurrence in the same way he does at the second and thus could have won in fewer moves than there are positions. [13]

This claim was challenged in 1927 by D. König [9]. After proving a very general theorem, he used it to (among other things) prove Zermelo’s claim. But König also argued in [9] that Zermelo’s original proof is incomplete because

Zermelo had argued that White could [change] his behavior at the first occurrence of any repeated winning position and thus win without repetition. [13]

The flaw of this argument, according to König, is that

Zermelo implicitly assumes that Black would never change his behavior at any reoccurrence of a winning position. He only considered the special case of unchanging behavior on Black’s part. What he needed to show was that his claim is true for all possible moves by Black. [13]

König communicated his results to Zermelo who accepted the criticism and provided a new, correct proof that the number of moves necessary to win is less than the number of positions.

Zermelo’s proof uses the *nontrivial*¹ result that the number of moves necessary to force a win is bounded [13];

this is the analog of Lemma 2.1. Zermelo’s proof is included in [9]. The following English translation originally appeared in [13].

Let p_0 be a position in which White—having to move—can force a checkmate however not in a bounded number of moves but, depending on the play of the opponent, in a possibly unbounded increasing number of moves. Then for every move by White, Black can bring about a position p_1 which has the same property. Otherwise, White could achieve his goal with a bounded number of moves starting from p_0 , as the number of possible moves is finite. Consequently, and independently of White’s play, if the opponent plays correctly, an unbounded sequence of positions p_0, p_1, p_2, \dots which all have the property [of] p_0 will emerge, i.e. which will never lead to a checkmate. Thus, if from a position p_0 a win can be forced at all, then it can be forced in a bounded number of moves. [13]

Clearly, the above argument can also be applied (for the case of CR) to prove Lemma 2.1.

4. MEMORYLESS STRATEGIES

We conjecture that Assumption (4) (and hence Lemma 2.1) appears self-evident because of an additional implicit assumption:

(5) “*in the CR game neither player loses anything by using memoryless strategies*”.

Assuming (5), we can prove that no game position is repeated (and hence capture time is bounded) by contradiction, as follows. If the cop uses a winning memoryless strategy and a position is repeated at rounds t_1 and t_2 , then the robber can repeat his moves between t_1 and $t_2 - 1$ and, since the cop’s strategy is memoryless, he will also repeat his moves. Hence at times $t_3 = t_2 + (t_2 - t_1)$, $t_4 = t_2 + 2(t_2 - t_1)$, ... the same position will be reached and the game will continue *ad infinitum*, which contradicts the assumption that the cop was using a winning strategy. But can we consider (5) self-evident? Informally it can be restated as follows:

(6) “*remembering how a CR position was reached gives no advantage to either player*”

¹Our emphasis.

and this seems quite reasonable. But in game theoretic terms, (5) states that there exist memoryless strategies σ_C^* , σ_R^* such that

$$(7) \quad T(\sigma_C^*, \sigma_R^*) = \sup_{s_R} \inf_{s_C} T(s_C, s_R) = \inf_{s_C} \sup_{s_R} T(s_C, s_R),$$

where the inf and sup are taken over *all* strategies (not just memoryless ones). We believe that (7) is far less obvious than either (5) or (6) and requires proof. The fact that neither (5) nor (6) was invoked by the authors of [9, 13, 14] (when studying the same issues for the game of chess) supports our position.

It appears that (5) is taken for granted in [7, 5, 6]. For example, it is stated in [7] that “the situation of the game can be described simply by saying where each player is and whose turn it is to move”, i.e., by the triple (x, y, p) ; this is reflected in the fact that in [5] (x, y, p) is called the *state* of the game. Also, in both [7, 5] the algorithm used to determine optimal strategies considers only memoryless strategies, implying that nothing is lost by ignoring strategies with memory, i.e., (5) is assumed *implicitly*. Finally, the definition of “strategy” used in [6] encompasses only memoryless strategies, which again suggests that the authors take (5) for granted.

Our own point of view is different. We believe that (5) must be proved, rather than assumed. The proof is easy but not trivial. Namely, (5) is a corollary of Theorem 2.2 which depends on Lemma 2.1; as already mentioned, this lemma can be proved by the previously mentioned argument from [9].

5. REACHABILITY GAMES

We can use another route, and establish (5) through well known results about *reachability games* [10]. This requires formulating CR as a *reachability game*; the connection is interesting and, as far as we know, has not been noted previously².

A reachability game [10] is played by two players (Player 0 and Player 1) on a *digraph* $\overline{G} = (\overline{V}, \overline{E})$. Each move consists in sliding a *token* from one digraph node to another, along an edge; the i -th player slides the token if and only if it is currently located on a node $v \in \overline{V}_i$ ($i \in \{0, 1\}$), where $\overline{V}_0 \cup \overline{V}_1 = \overline{V}$, $\overline{V}_0 \cap \overline{V}_1 = \emptyset$. Player 0 wins if and only if the token goes into a node $u \in \overline{F}$; otherwise Player 1 wins. The game is fully described by the tuple $(\overline{V}_0, \overline{V}_1, \overline{E}, \overline{F})$. The following is well known [3, 10].

Theorem 5.1. *Let $(\overline{V}_0, \overline{V}_1, \overline{E}, \overline{F})$ be a reachability game on the digraph $\overline{D} = (\overline{V}, \overline{E})$. Then \overline{V} can be partitioned into two sets \overline{W}_0 and \overline{W}_1 such that (for $i \in \{0, 1\}$) player i has a memoryless strategy σ_i which is winning whenever the game starts in $u \in \overline{W}_i$.*

CR can be converted to a reachability game. Essentially, this has been done in [7, 5] (even though the authors appear to not be aware of the connection to reachability games) using the *move digraph* M_G . Every node of M_G corresponds to a *position* (x, y, p) , where x (resp. y) is the cop (resp. robber) position in the original G and p is the player whose turn it is to play; (u, v) is a directed edge of M_G iff it is possible to get from u to v by a single move. The M_G thus constructed can be used to play any *modified* CR game with prespecified initial player positions x and y and starting player p . For the “classic” CR game (starting on an “empty board”) M_G must be expanded to \overline{M}_G by adding:

- (1) one node of the form $(\emptyset, \emptyset, C)$ (it corresponds to the beginning of the game, just before the cop is placed in the graph);

²Despite the fact that the reachability formulation is almost identical to the one used in [7, 5].

- (2) $|V|$ nodes of the form (x, \emptyset, C) with $x \in V$ (they correspond to the middle of the 0-th round, when the cop has been placed but not the robber);
- (3) the edges which correspond to legal moves between the nodes of \overline{M}_G .

Let $\overline{M}_G = (\overline{V}, \overline{E})$ and consider the reachability game $(\overline{V}_0, \overline{V}_1, \overline{E}, \overline{F})$ where \overline{V}_0 (resp. \overline{V}_1) contains the nodes of the form (x, y, C) (resp. (x, y, R)) and \overline{F} (the cop's *target set*) contains the nodes of the form (x, x, p) , i.e., positions of the CR game in which the cop and robber are in the same position (i.e., node of the original G). This reachability game subsumes both the classic and the (previously mentioned) modified CR games.

The graph G is cop-win iff $(\emptyset, \emptyset, C)$ belongs to the \overline{W}_0 ; in this case, by Theorem 5.1, the cop has a memoryless winning strategy for the classic CR game. To establish that he has a *time optimal* memoryless strategy, we use the fact that the memoryless winning strategy yields bounded capture time; hence, by the arguments of Section 2, both cop and robber have memoryless time optimal strategies. The situation is similar when G is robber-win. In this case, $(\emptyset, \emptyset, C) \in \overline{V}_1$ and, by Theorem 5.1, the robber has a memoryless winning strategy σ_R^* and, since the capture time is infinite, *any* winning robber strategy is also time optimal. When the robber uses σ_R^* , any *cop* strategy results in infinite capture time; hence any memoryless strategy σ_C is time optimal. These ideas can be extended to any graph G and any number K of cops, provided the move digraph $\overline{M}_G^{(K)} = (\overline{V}^{(K)}, \overline{E}^{(K)})$ is constructed accordingly. The *cop number* $c(G)$ is the smallest K such that $(\emptyset, \emptyset, C) \in \overline{V}_0^{(K)}$.

We have already mentioned that the move digraph formulation of CR has appeared in [7, 5]. It has also been used in the early paper [2] to study winning strategies (but *not* time optimality); the authors of [2] appear unaware of the connection to reachability games.

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